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On the steady solutions of the problem of Rayleigh–Taylor instability

By M. J. TAN†
Centre d'Etudes Nucléaires de Grenoble, Service des Transferts Thermiques,
85 X, 38041 Grenoble, France

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The problem of Rayleigh–Taylor instability is reexamined within the framework of incompressible, inviscid and irrotational fluid flow in a bounded three-dimensional domain. A relation proposed by Pimbley (1976) between the slope and the amplitude of the interface at the rigid boundary is adopted as the interface boundary condition. Steady solutions are derived in approximate form by using bifurcation theory. It is shown that under the conditions given some of the steady solutions exhibit the features of the well-known bubbles-and-spikes configuration and can be stable to infinitesimal disturbances.

1. Introduction

Rayleigh–Taylor instability refers to the instability of the accelerated interface between two fluids of different densities, which occurs when the acceleration is directed from the lighter fluid to the heavier fluid. Since it was investigated first by Rayleigh (1900) and later by Taylor (1950), this instability problem has been the subject of numerous studies (e.g. Lewis 1950; Bellman & Pennington 1954; Emmons, Chang & Watson 1960; Miles & Dienes 1966; Daly 1969; Pimbley 1976; Dienes 1978; Baker, Meiron & Orszag 1980; Baker & Freeman 1981; Menikoff & Zemach 1983).

Most of these studies are concerned with the evolution of initially sinusoidal two-dimensional disturbances of a plane interface. The amplification of such disturbances is experimentally discussed in terms of three stages (Lewis 1950; Emmons *et al.* 1960):

(i) an initial stage during which the amplitudes of the disturbances increase exponentially;

(ii) a transition stage during which the sinusoidal form of the disturbances is lost; the interface changes to the shape of rising broad rounded-ended columns (bubbles) of the lighter fluid and falling narrow columns (spikes) of the heavier fluid;

(iii) a final asymptotic stage during which the round-ended columns of the lighter fluid rise through the heavier fluid at a constant velocity; the shape of the interface remains unchanged.

The phenomena associated with stage (i) are well predicted by linear theories (Bellman & Pennington 1954; Miles & Dienes 1966), whereas the nonlinear phenomena associated with stages (ii) and (iii) are incompletely understood. On the one hand, the change of growth rate from exponential in time to algebraic in time implies the existence of some nonlinear bound on the growth rate. Indeed, this is borne out by

† Present address: Chemical Engineering Department, Northwestern University, Evanston, Illinois 60201.

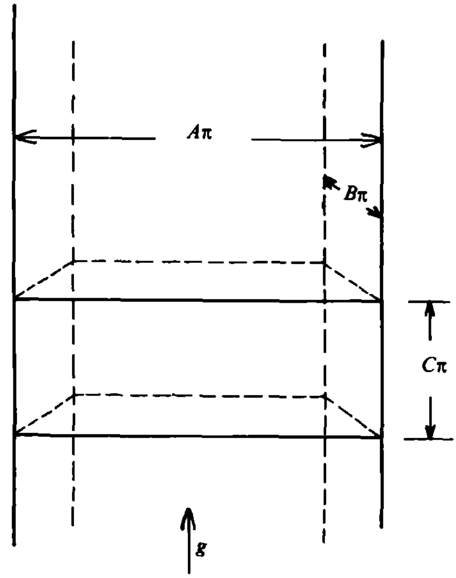

FIGURE 1. Schematic diagram of the physical system under consideration.

computer simulations (Daly 1969; Baker *et al.* 1980; Menikoff & Zemach 1983). Recent theoretical studies (Dienes 1978; Baker & Freeman 1981) have also found some success in estimating the nonlinear bound.

On the other hand, the development of the interface during stage (iii) suggests the existence of some new stable modes of equilibrium. However, this aspect of the problem of nonlinear Rayleigh–Taylor instability has scarcely been discussed. The question thus arises: are there any equilibrium states of the interface that are stable to infinitesimal disturbances? To answer this question we must consider the instability problem in the light of physical systems. Though the experiments were carried out with finite apparatus, most previous theoretical work has considered fluids of infinite lateral extent, thereby obviating the boundary conditions that govern the motion of fluid where the interface meets the rigid boundary. This naturally leads to another question: does the presence of a rigid boundary have any stabilizing effects on the interface?

The paper of Pimbley (1976) shed some light on these two questions. The two-dimensional problem of Rayleigh–Taylor instability together with a proposed interface boundary condition was studied as an evolution problem in nonlinear partial differential equations in the large. The existence and nature of the steady solutions that bifurcate from the given one, i.e. the quiescent plane interface, and the stability of the bifurcating solutions were examined in a rigorous manner. It was found that some of the new steady solutions do have an appearance suggestive of the shape of the interface during stage (iii), but none of them is stable to infinitesimal disturbances.

In the present paper we adopt Pimbley's approach to study the three-dimensional problem of Rayleigh–Taylor instability. Section 2 contains a general description of the problem, which is formulated within the framework of incompressible, inviscid and irrotational fluid flow in a bounded three-dimensional domain. In §3 nontrivial steady solutions are derived in approximate form through a rather straightforward application of bifurcation theory. Section 4 contains a numerical example and some concluding remarks.

2. Formulation

Consider a physical system, as depicted in figure 1, in which a heavier fluid of depth $C\pi$ is confined by two lighter fluids of infinite depth and the walls of a vertical rectangular vessel of length $A\pi$ and width $B\pi$. We suppose that the fluids are motionless and that the interfaces are horizontal and flat. At time $t = 0$ an impulsive pressure change is applied to the lower interface. We wish to study the subsequent motions of the lower interface and the heavier fluid under the following assumptions:

(*a*) the densities of the lighter fluids are negligibly small;
(*b*) there is no mass transfer at the fluid interfaces;
(*c*) the heavier fluid is inviscid and incompressible;
(*d*) the flow of the heavier fluid is irrotational.

We choose a Cartesian coordinate system that has its origin fixed at the centre of the upper interface and is oriented so that the x-axis and the y-axis are parallel to the walls of the vessel and the unit vector associated with the z-axis is parallel to and of the same sense as the direction of the apparent acceleration. The upper and lower interfaces can then be described respectively by

$$S_1(x, y, z, t) = z = 0, \tag{1}$$

$$S_2(x, y, z, t) = z + C\pi - \eta(x, y, t) = 0. \tag{2}$$

Equation (1) represents a flat upper interface and is in accordance with the well-known experimental observations that the interface remains flat when the acceleration is directed from the heavier fluid toward the lighter fluid (Lewis 1950).

Following the steps used by Emmons *et al.* (1960), we arrive at the following set of equation and conditions:

continuity equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad (-\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi, S_1 < 0, S_2 > 0); \tag{3}$$

boundary conditions

$$\phi_z = 0 \quad (-\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi, S_1 = 0); \tag{4}$$

$$\left.\begin{aligned} -\eta_t - \eta_x\phi_x - \eta_y\phi_y + \phi_z &= 0 \\ \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta + \frac{2\sigma H}{\rho} &= 0 \end{aligned}\right\} \quad \begin{aligned} &(-\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi, \\ &\qquad -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi, S_2 = 0); \end{aligned} \quad \begin{cases} (5) \\ (6) \end{cases}$$

initial condition

$$\eta(x, y, 0) \text{ is prescribed} \quad (-\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi), \tag{7}$$

where a subscript indicates differentiation with respect to that variable, ϕ is the velocity potential, g is the apparent acceleration, ρ is the density of the heavier fluid, σ is the surface tension between the heavier fluid and the lower lighter fluid, and H is the mean curvature given by (Korn & Korn 1961)

$$H = \frac{(1+\eta_y^2)\,\eta_{xx} - 2\eta_x\,\eta_y\,\eta_{xy} + (1+\eta_x^2)\,\eta_{yy}}{2(1+\eta_x^2+\eta_y^2)^{\frac{3}{2}}}. \tag{8}$$

For fluids of infinite extent the interface conditions (4)–(6) together with the continuity equation (3) are sufficient to solve the problem. Inasmuch as we are concerned with the motions of the fluid and the interface in a finite domain, we need to specify additional boundary conditions.

From potential theory we have

$$\phi_x = 0 \quad (x = \pm\tfrac{1}{2}A\pi, S_1 < 0, S_2 > 0), \tag{9}$$

$$\phi_y = 0 \quad (y = \pm\tfrac{1}{2}B\pi, S_1 < 0, S_2 > 0). \tag{10}$$

The appropriate boundary conditions that govern the motion of fluid in the vicinity of a moving contact line are as yet unknown (Dussan V. 1983). On the other hand, the steady problem entails the Helmholtz equation for η, as will be seen in §3. It is well established (Morse & Feshbach 1953, p. 706) that the association of Dirichlet or Neumann boundary conditions with an elliptic equation leads to a unique solution. Hence it is not unreasonable to specify the value of η, the amplitude, at the wall or the value of the normal gradient of η, the slope of the interface, there. Pimbley (1976) remarked that 'the question is less what boundary conditions would be given by the local physics at the wall, and more what boundary conditions would produce a physically reasonable and observed solution away from the wall'. In the same spirit, we assume that the slope of the interface at the wall depends only on the amplitude of the interface there. Let α_+, α_-, β_+, β_- be functions of η; we thus have

$$\eta_x = \alpha_+(\eta) \quad (x = \tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi), \tag{11}$$

$$\eta_x = \alpha_-(\eta) \quad (x = -\tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi), \tag{12}$$

$$\eta_y = \beta_+(\eta) \quad (y = \tfrac{1}{2}B\pi,\ -\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi), \tag{13}$$

$$\eta_y = \beta_-(\eta) \quad (y = -\tfrac{1}{2}B\pi,\ -\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi). \tag{14}$$

Given that the boundary is symmetric, we impose the following conditions:

$$\alpha_+(\eta(\tfrac{1}{2}A\pi, y, t)) = -\alpha_-(\eta(-\tfrac{1}{2}A\pi, y, t)) \quad \text{if } \eta(\tfrac{1}{2}A\pi, y, t) = \eta(-\tfrac{1}{2}A\pi, y, t), \tag{15}$$

$$\beta_+(\eta(x, \tfrac{1}{2}B\pi, t)) = -\beta_-(\eta(x, -\tfrac{1}{2}B\pi, t)) \quad \text{if } \eta(x, \tfrac{1}{2}B\pi, t) = \eta(x, -\tfrac{1}{2}B\pi, t). \tag{16}$$

Without loss of generality, we suppose

$$\alpha_+(\eta) = f(\eta) \quad (x = \tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi), \tag{17}$$

$$\beta_+(\eta) = f(\eta) \quad (y = \tfrac{1}{2}B\pi,\ -\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi), \tag{18}$$

where f is a yet-to-be-specified function of η. Our hypothesis that the quiescent interface is flat implies $f(0) = 0$. At the same time, we expect that the slope of the interface at the wall approaches a constant as the amplitude becomes large and that the value of the constant depends on some physical constants dictated by the local physics at the wall.

Thus we can rewrite the boundary conditions (11)–(14) as

$$\eta_x = f(\eta) \quad (x = \tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi), \tag{19}$$

$$\eta_x = -f(\eta) \quad (x = -\tfrac{1}{2}A\pi,\ -\tfrac{1}{2}B\pi < y < \tfrac{1}{2}B\pi), \tag{20}$$

$$\eta_y = f(\eta) \quad (y = \tfrac{1}{2}B\pi,\ -\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi), \tag{21}$$

$$\eta_y = -f(\eta) \quad (y = -\tfrac{1}{2}B\pi,\ -\tfrac{1}{2}A\pi < x < \tfrac{1}{2}A\pi), \tag{22}$$

where the function f satisfies

$$f(\eta) = 0 \quad (\eta = 0), \tag{23a}$$

$$f(\eta) \to -F \quad (\eta \to \pm\infty), \tag{23b}$$

$$0 > f(\eta) > -F \quad (\eta \neq 0). \tag{23c}$$

Moreover, we assume that $f'(0) = 0$ and that in the neighbourhood of $\eta = 0$, f can be represented by a MacLaurin series:

$$f(\eta) = \sum_{i=2}^{\infty} f_i \eta^i. \tag{24}$$

Introducing into (3)–(7), (9)–(10) and (19)–(22) scaled variables defined by

$$\left.\begin{aligned} &x^* \triangleq \frac{x}{A}, \quad y^* \triangleq \frac{y}{B}, \quad z^* \triangleq \frac{z}{C}, \quad \eta^* \triangleq \frac{\eta}{C}, \\ &t^* \triangleq \frac{tg^{\frac{1}{2}}}{C^{\frac{1}{2}}}, \quad \phi^* \triangleq \frac{\phi}{g^{\frac{1}{2}}C^{\frac{3}{2}}}, \quad f^*(\eta^*) \triangleq f(C\eta^*), \\ &S_i^*(x^*, y^*, z^*, t^*) = S_i\left(Ax^*, By^*, Cz^*, \frac{C^{\frac{1}{2}}t^*}{g^{\frac{1}{2}}}\right) \quad (i = 1, 2), \end{aligned}\right\} \tag{25}$$

we obtain the following:

continuity equation

$$\mathrm{L}\phi + \phi_{zz} = 0 \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_1 < 0, S_2 > 0); \tag{26}$$

boundary conditions

$$\phi_x = 0 \quad (x = \pm\tfrac{1}{2}\pi, S_1 < 0, S_2 > 0), \tag{27}$$

$$\phi_y = 0 \quad (y = \pm\tfrac{1}{2}\pi, S_1 < 0, S_2 > 0), \tag{28}$$

$$\phi_z = 0 \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_1 = 0), \tag{29}$$

$$-\eta_t - a^2\eta_x\phi_x - b^2\eta_y\phi_y + \phi_z = 0 \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_2 = 0), \tag{30}$$

$$\phi_t + \tfrac{1}{2}(a^2\phi_x^2 + b^2\phi_y^2 + \phi_z^2) + \eta + \frac{1}{\lambda}\frac{\mathrm{L}\eta + a^2b^2(\eta_y^2\,\eta_{xx} - 2\eta_x\,\eta_y\,\eta_{xy} + \eta_x^2\,\eta_{yy})}{(1 + a^2\eta_x^2 + b^2\eta_y^2)^{\frac{3}{2}}} = 0$$

$$(-\tfrac{1}{2}p < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_2 = 0), \tag{31}$$

$$\eta_x = \frac{1}{a}f(\eta) \qquad (x = \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \tag{32}$$

$$\eta_x = -\frac{1}{a}f(\eta) \quad (x = -\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \tag{33}$$

$$\eta_y = \frac{1}{b}f(\eta) \qquad (y = \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi), \tag{34}$$

$$\eta_y = -\frac{1}{b}f(\eta) \quad (y = -\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi); \tag{35}$$

initial condition

$$\eta(x, y, 0) \text{ is prescribed} \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi). \tag{36}$$

Asterisks have been suppressed for convenience,

$$a \triangleq \frac{C}{A}, \quad b \triangleq \frac{C}{B}, \quad \lambda \triangleq \frac{\rho g C^2}{\sigma}, \tag{37}$$

and L is a second-order linear operator:

$$\mathrm{L} \triangleq a^2\frac{\partial^2}{\partial x^2} + b^2\frac{\partial^2}{\partial y^2}. \tag{38}$$

To recapitulate, we consider the motions of the heavier fluid and the lower interface that are governed by the system of equations (26)–(36).

3. Steady solutions

The steady problem is obtained by setting $\eta_t = 0$ and $\phi_t = 0$ in (30) and (31). It is obvious that the initial state, $\eta = 0$ and $\phi = \text{const}$, satisfies the resulting equations. It should be noted that $\eta = 0$ would not be a steady solution were the interface boundary function $f(\eta)$ in the boundary conditions (32)–(35) chosen identically to be the fixed contact angle between the heavier fluid and the rigid boundary. This flat quiescent interface will hereinafter be referred to as the basic solution. Our goal in this section is to seek other steady solutions that bifurcate from the basic solution.

We begin by showing that under steady condition ϕ is constant in the bounded flow region. From Green's theorem (Morse & Feshbach 1953, p. 803), we have

$$\int_R |\nabla\phi|^2 \,\mathrm{d}V = -\int_R \phi\nabla^2\phi \,\mathrm{d}V + \int_{\partial R_1} \phi\phi_x \,\mathrm{d}A + \int_{\partial R_2} \phi\phi_y \,\mathrm{d}A + \int_{\partial R_3} \phi\phi_z \,\mathrm{d}A + \int_{\partial R_4} \phi\nabla\phi\cdot\boldsymbol{n} \,\mathrm{d}A, \quad (39)$$

in which the volume R and the bounding surfaces ∂R_1, ∂R_2, ∂R_3 and ∂R_4 are defined as follows:

$$R = \triangleq \{(x,y,z) \,|\, -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_1 < 0, S_2 > 0\}, \quad (40a)$$

$$\partial R_1 \triangleq \{(x,y,z) \,|\, x = \pm\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_1 < 0, S_2 > 0\}, \quad (40b)$$

$$\partial R_2 \triangleq \{(x,y,z) \,|\, y = \pm\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi, S_1 < 0, S_2 > 0\}, \quad (40c)$$

$$\partial R_3 \triangleq \{(x,y,z) \,|\, -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_1 = 0\}, \quad (40d)$$

$$\partial R_4 \triangleq \{(x,y,z) \,|\, -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi, S_2 = 0\}. \quad (40e)$$

The terms on the right-hand side of (39) vanish by virtue of (26)–(30). Hence $\nabla\phi$ vanishes, or $\phi = \text{const}$, identically in R.

With $\phi = \text{const}$ the steady problem reduces to a nonlinear problem in η only:

$$\mathrm{L}\eta + \lambda\eta(1 + a^2\eta_x^2 + b^2\eta_y^2)^{\frac{3}{2}} + a^2b^2(\eta_y^2\,\eta_{xx} - 2\eta_x\,\eta_y\,\eta_{xy} + \eta_x^2\,\eta_{yy}) = 0$$
$$(-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \quad (41a)$$

$$\eta_x = \frac{1}{a}f(\eta) \qquad (x = \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \quad (41b)$$

$$\eta_x = -\frac{1}{a}f(\eta) \quad (x = -\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \quad (41c)$$

$$\eta_y = \frac{1}{b}f(\eta) \qquad (y = \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi), \quad (41d)$$

$$\eta_y = -\frac{1}{b}f(\eta) \quad (y = -\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi). \quad (41e)$$

It can be shown (Stakgold 1971) that the bifurcation points of the nonlinear problem (41 *a–e*) are to be found among the eigenvalues of the linearized (about the basic solution) problem:

$$\mathrm{L}\eta + \lambda\eta = 0 \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \quad (42a)$$

$$\eta_x = 0 \quad (x = \pm\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \quad (42b,\ c)$$

$$\eta_y = 0 \quad (y = \pm\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi), \quad (42d,\ e)$$

and that if an eigenvalue is simple, one and only one non-trivial solution bifurcates from the basic solution at that point.

The eigenvalues λ_{mn} and the corresponding eigenfunctions η_{mn} for the linearized problem (42*a–e*), are, for m, $n = 1, 2, \ldots$, readily found to be

$$\lambda_{mn} = (am)^2 + (bn)^2 \tag{43}$$

and

$$\eta_{mn} = \begin{cases} \cos mx \cos ny & (m, n \text{ even}), \\ \cos mx \sin ny & (m \text{ even}, n \text{ odd}), \\ \sin mx \cos ny & (m \text{ odd}, n \text{ even}), \\ \sin mx \sin ny & (m, n \text{ odd}). \end{cases} \tag{44}$$

If the aspect ratio b/a is not an integer then the eigenvalues are simple. Otherwise, the eigenvalues can be simple or multiple depending on the values of m and n. For example, if $a = b$ then

$\lambda_{11} = 2a$ is a simple eigenvalue,

$\lambda_{12} = \lambda_{21} = 5a$ is an eigenvalue of multiplicity two,

$\lambda_{55} = \lambda_{17} = \lambda_{71} = 50a$ is an eigenvalue of multiplicity three,

$\lambda_{47} = \lambda_{74} = \lambda_{18} = \lambda_{81} = 65a$ is an eigenvalue of multiplicity four.

It has been suggested (Bauer, Keller & Reiss 1975) that secondary bifurcation can arise in the neighbourhood of multiple eigenvalues provided that a splitting parameter exists. As the parameters a and b are intrinsic to the current geometry, the case in which the eigenvalues are multiple appears to be a plausible candidate for further study on secondary bifurcation, which can be a research subject in its own right. In what follows we confine ourselves to the case of simple eigenvalues.

Now that we know that the basic solution does branch out into non-trivial solutions and where the bifurcation points are, we can construct the bifurcation solution $\bar{\eta}$ near a bifurcation point λ_{mn} by means of nonlinear perturbation theory. Introducing an artificial small parameter ϵ, we seek asymptotic series expansions of the solutions to (41*a–e*) in the form

$$\bar{\eta}(x,y) \sim \sum_{i=1}^{\infty} \eta_{mn}^{(i)}(x,y)\,\epsilon^i, \tag{45}$$

$$\bar{\lambda} - \lambda_{mn} \sim \sum_{i=1}^{\infty} \lambda_{mn}^{(i)}\,\epsilon^i. \tag{46}$$

The small parameter ϵ can be regarded as a measure of the amplitude. We define ϵ for definiteness as

$$\epsilon \triangleq \langle \bar{\eta}, \eta_{mn}^{(1)} \rangle \tag{47}$$

where the angle brackets denote the inner product

$$\langle u, v \rangle \triangleq \left(\frac{2}{\pi}\right)^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} uv \,\mathrm{d}x\,\mathrm{d}y. \tag{48}$$

Substituting (45)–(46) into (41$a-e$) and equating like powers of ϵ in the resulting expressions, we obtain a sequence of linear problems. They are given, for $j = 1, 2, \ldots$, by

$$\mathrm{L}\eta_{mn}^{(j)} + \lambda_{mn}\eta_{mn}^{(j)} = R^{(j)} \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \tag{49a}$$

$$\eta_{mn,x}^{(j)} = B_1^{(j)} \quad (x = \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \tag{49b}$$

$$\eta_{mn,x}^{(j)} = B_2^{(j)} \quad (x = -\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi), \tag{49c}$$

$$\eta_{mn,y}^{(j)} = B_3^{(j)} \quad (y = \tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi), \tag{49d}$$

$$\eta_{mn,y}^{(j)} = B_4^{(j)} \quad (y = -\tfrac{1}{2}\pi,\ -\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi), \tag{49e}$$

where $R^{(j)}$, $B_1^{(j)}$, $B_2^{(j)}$, $B_3^{(j)}$ and $B_4^{(j)}$ are functions of $\lambda_{mn}^{(i)}$ and $\eta_{mn}^{(i)}$ for $i < j$, and are given, for j up to 3, in the Appendix. In particular,

$$R^{(1)} = B_1^{(1)} = B_2^{(1)} = B_3^{(1)} = B_4^{(1)} = 0.$$

Hence (49a–e) for $j = 1$ correspond to the homogeneous linearized problem (42a–e), which has the solution $\eta_{mn}(x, y)$. Using the definition (47) of ϵ, we find that

$$\eta_{mn}^{(1)}(x, y) = \eta_{mn}(x, y). \tag{50}$$

For $j = 2, 3, \ldots$, (49a–e) are inhomogeneous and therefore are solvable only if $R^{(j)}$ and $B_1^{(j)}$, $B_2^{(j)}$, $B_3^{(j)}$, $B_4^{(j)}$ satisfy the following consistency condition (Stakgold 1979):

$$\langle R^{(j)}, \eta_{mn}\rangle = \left(\frac{2a}{\pi}\right)^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} [B_1^{(j)}\eta_{mn}|_{x=\frac{1}{2}\pi} - B_2^{(j)}\eta_{mn}|_{x=-\frac{1}{2}\pi}]\,\mathrm{d}y + \left(\frac{2b}{\pi}\right)^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} [B_3^{(j)}\eta_{mn}|_{y=\frac{1}{2}\pi} - B_4^{(j)}\eta_{mn}|_{y=-\frac{1}{2}\pi}]\,\mathrm{d}x. \tag{51}$$

By analysing the system of equations (49a–e) and (51) in sequence of j, one can calculate the coefficients in (45) and (46). The calculations for $j = 2, 3$ are elementary but lengthy. We therefore omit the details and summarize the results as follows.

(i) $j = 2$

The consistency condition (51) gives

$$\lambda_{mn}^{(1)} = 0. \tag{52}$$

With $\lambda_{mn}^{(1)} = 0$, $\eta_{mn}^{(2)}(x, y)$ is found to be

$$\eta_{mn}^{(2)}(x, y) = \frac{f_2}{2a}\left\{\frac{\cos(\alpha_1 x)}{\alpha_1 \sin(\frac{1}{2}\alpha_1 \pi)} + \frac{\cos\left(\frac{a}{b}\alpha_1 y\right)}{\alpha_1 \sin\left(\frac{a}{b}\frac{\alpha_1 \pi}{2}\right)} + (-1)^n \cos(2ny)\frac{\cos(\alpha_2 x)}{\alpha_2 \sin(\frac{1}{2}\alpha_2 \pi)} + (-1)^m \cos(2mx)\frac{\cos\left(\frac{a}{b}\alpha_3 y\right)}{\alpha_3 \sin\left(\frac{a}{b}\frac{\alpha_3 \pi}{2}\right)}\right\}, \tag{53}$$

where
$$\alpha_1 = \left[m^2+\left(\frac{b}{a}\right)^2 n^2\right]^{\frac{1}{2}}, \tag{54a}$$
$$\alpha_2 = \left[m^2-3\left(\frac{b}{a}\right)^2 n^2\right]^{\frac{1}{2}}, \tag{54b}$$
$$\alpha_3 = \left[\left(\frac{b}{a}\right)^2 n^2-3m^2\right]^{\frac{1}{2}}. \tag{54c}$$

(ii) $j = 3$

The consistency condition gives
$$\lambda^{(2)}_{mn} = -c_1+c_2 f_2^2-c_3 f_3, \tag{55}$$
where
$$c_1 = a^4\left\{\frac{9}{32}\left[m^4+\left(\frac{b}{a}\right)^4 n^4\right]+\frac{1}{16}\left(\frac{b}{a}\right)^2 m^2n^2\right\}, \tag{56a}$$
$$c_2 = \frac{b}{a}\left\{\left[\frac{a}{b}\frac{\cot(\frac{1}{2}\alpha_1\pi)}{\alpha_1}+\frac{\cot\left(\frac{a}{b}\frac{\alpha_1\pi}{2}\right)}{\alpha_1}+\frac{a}{b}\frac{\cot(\frac{1}{2}\alpha_2\pi)}{2\alpha_2}+\frac{\cot\left(\frac{a}{b}\frac{\alpha_3\pi}{2}\right)}{2\alpha_3}\right]\pi+4\left(\frac{2}{3}\frac{1}{\alpha_1^2}+\frac{1}{\alpha_2^2}+\frac{1}{\alpha_3^2}\right)\right\}, \tag{56b}$$
$$c_3 = \frac{3\pi}{4}a\left(1+\frac{b}{a}\right). \tag{56c}$$

While the values of c_1 and c_3 are always positive, the value of c_2 can be positive or negative depending on the aspect ratio b/a and the values of m and n. Therefore, for a given geometry and a given interface boundary function f, $\lambda^{(2)}_{mn}$ may be positive, zero or negative, depending on the particular bifurcating solution under consideration.

In the case of $\lambda^{(2)}_{mn} = 0$, one needs to solve for $\eta^{(3)}_{mn}(x,y)$ and then carry out the calculations for $j = 4$. The analysis is similar in principle but conceivably more tedious. Hence we do not consider this pathological case and proceed to describe some qualitative features of the solution $\bar{\eta}$ in the neighbourhood of the bifurcation point λ_{mn}.

To the order of the first non-vanishing terms we have
$$\bar{\eta} \sim \eta^{(1)}_{mn}\,\epsilon,$$
$$\bar{\lambda}-\lambda_{mn} \sim \lambda^{(2)}_{mn}\,\epsilon^2.$$
Thus in the $(\bar{\lambda}, \|\bar{\eta}\|)$-plane a parabolic branch would emanate from $\bar{\lambda} = \lambda_{mn}$. The parabola points to the left if $\lambda^{(2)}_{mn}$ is negative; it points to the right if $\lambda^{(2)}_{mn}$ is positive. A typical bifurcation diagram is sketched in figure 2, where only one bifurcating solution is shown for clarity.

4. Discussion

In §3 the nonlinear problem governed by (26)–(36) has been studied with the aid of bifurcation theory. The initial state is a basic solution that holds for all values of λ. The bifurcation points at which non-trivial steady solutions join the basic solution have been located. Qualitative information about the shape of these bifurcating

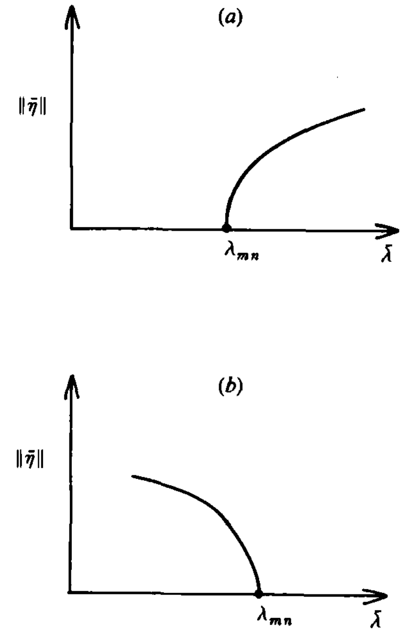

FIGURE 2. Bifurcation diagrams: (a) supercritical bifurcation, $\lambda^{(2)}_{mn} > 0$; (b) subcritical bifurcation, $\lambda^{(2)}_{mn} < 0$.

solutions near their intersections with the basic solution has been obtained. The bifurcating solutions point to the left or to the right, depending on the locations of the bifurcation points as well as on the hypothetical interface boundary function. Moreover, it can be shown with standard linear stability analysis that the basic solution is stable to infinitesimal disturbances for λ in the range $0 < \lambda < \lambda_{11}$ and unstable outside this range, and that the right-pointing bifurcating solutions are stable whereas the left-pointing ones are unstable.

We now turn to the physical implications of the preceding analysis. In case of $\lambda > \lambda_{11}$, the physical system can no longer stay in the initial state. If all bifurcating solutions are left-pointing then the steady states that can be obtained from bifurcation theory are all unstable. If there exist some right-pointing solutions then the physical system may evolve into one or more of the new steady states represented by those solutions. Pimbley's (1976) work on a one-dimensional interface confined between two parallel walls shows that all bifurcating solutions are left-pointing, whereas the present work on a two-dimensional interface confined by four walls shows that, with similar boundary conditions, there can be right-pointing bifurcating solutions. The present results thus imply that the presence of rigid boundaries introduces a stabilizing effect on the shape of the interface.

The contrast between the present results and those of Pimbley can be interpreted in the light of experimental observations. The most common arrangement for Rayleigh–Taylor experiments involves a rectangular container of liquid. Emmons *et al.* (1960) reported that there are liquid films at each wall of the rectangular container. The photographic sequences in Lewis's (1950) paper show that the magnitude of the initial acceleration has virtually no influence upon the number of round-ended columns of the lighter fluid observed during stages (ii) and (iii). We infer from these observations that the problem of Rayleigh–Taylor instability is truly three-dimensional and consequently that the development of the interface in general and the existence of new stable modes of equilibrium in particular are affected by the presence of all four walls. To the extent that the stable steady bifurcating solutions can be regarded as the shapes of the interface that become dominant at later stages, it is thus physically reasonable that the bifurcating solutions in the present three-dimensional approach can have a stability that is absent when the work is confined to two dimensions. It is worthwhile to note that Pimbley's results cannot be recovered from the results presented in this paper by setting one of the two aspect ratios a or b equal to zero. This is due to the fact that two of the four boundary conditions (32)–(35) would become singular if a or b was set equal to zero, unless the interface boundary function f was chosen to be identically zero.

To illustrate the main points of the present work, we shall consider a numerical example. We first note that for a given pair of fluids the value of the parameter λ is controlled by varying the depth of the heavier fluid or the pressure applied to the lower interface, and thereby the acceleration. While it is not difficult to consider numerous experimental schemes that have various boundary geometries and initial conditions, we limit our discussions to the laboratory arrangement pertaining to film 37 of Lewis (1950, figure 10), in which the aspect ratio of the test section is 5 and the depth of the heavier fluid is of the same order of magnitude as the width of the test section. The values of the pertinent parameters are

$$\rho = 998\ \mathrm{kg/m^3}, \quad \sigma = 7.28\times 10^{-2}\ \mathrm{N/m}, \quad g = 351\ \mathrm{m/s^2},$$
$$A = 202\ \mathrm{mm}, \quad B = 40.4\ \mathrm{mm}, \quad C = 50.1\ \mathrm{mm}.$$

It follows immediately that

$$a = 0.248, \quad b = 1.24, \quad c = 121.$$

Since λ exceeds the first critical value

$$\lambda_{11} = 1.60,$$

the initial state is unstable. One can proceed further if the specific form of the interface boundary function f is known. To fix ideas we consider the function

$$f(\eta) = F[\exp(-\eta^2)-1]. \tag{57}$$

This even function automatically meets the hypothesized qualifications on the interface boundary functions (23a–c). Owing to the evenness of f, the third term on the right-hand side of (55) vanishes. Thus

$$\lambda^{(2)}_{mn} = -c_1 + c_2 F^2. \tag{58}$$

We have determined that, among the non-trivial steady states, when subjected to infinitesimal disturbances, those with $\lambda^{(2)}_{mn} < 0$ are unstable whereas those with $\lambda^{(2)}_{mn} > 0$ are stable. It is clear from (58) that $\lambda^{(2)}_{mn}$ can be positive only if $c_2 > 0$ and $c_1/c_2 < F^2$. Therefore, if all c_2 were negative then, without concerning ourselves with the physical meaning of F, we could infer that there existed no stable steady states. This is not the case. Table 1 gives all the (m,n)-pairs for which m and n are even, the eigenvalues λ_{mn} given by (43) are simple and less than λ, and the values of c_2 given by (56b) are positive. It can be seen that there are 22 possibly stable steady states.

To see how many and which of the 22 steady states are stable, it is necessary that the value of the constant F in (57) be specified. As our purpose is not to verify the

m	n	λ_{mn}	c_1	c_2	θ_{mn}
42	2	114.64	3362.9	0.21661	0.46°
38	4	113.41	2525.1	0.09349	0.35°
30	6	110.71	1915.0	0.24433	0.65°
14	8	110.46	2838.6	0.89484	1.02°
12	8	107.26	2800.1	0.47632	0.75°
36	4	104.31	2079.7	0.54601	0.93°
28	6	103.57	1682.5	0.87094	1.30°
8	8	102.34	2752.1	1.3243	1.26°
32	2	69.13	1150.4	0.02946	0.29°
26	4	66.18	720.33	0.12492	0.75°
12	6	64.21	914.46	1.3915	2.23°
24	4	60.03	577.67	0.87183	2.23°
8	6	59.29	879.73	1.3131	2.21°
6	6	57.57	870.79	2.3739	2.99°
22	2	35.92	271.31	0.42234	2.26°
8	4	28.54	180.63	1.3820	4.50°
6	4	26.82	175.01	2.3602	6.62°
18	2	26.08	129.98	4.8009	10.88°
4	4	25.59	172.01	5.3777	10.03°
10	2	12.30	23.642	0.48541	8.15°
4	2	7.13	11.290	5.4470	34.78°
2	2	6.40	10.751	21.632	54.82°

TABLE 1. Calculated values of λ_{mn}, c_1, c_2 and θ_{mn}

hypothesis that the particular interface boundary function given by (57) exists, but rather to show how the principles derived in a fairly general manner may be applied to particular problems, we *assume* in addition that F is given by

$$F = \cot\theta_c, \tag{59}$$

where θ_c is the contact angle between the heavier fluid and the rigid boundary.

Substituting (59) in (57) and letting

$$\theta_{mm} \triangleq \cot^{-1}\left(\frac{c_1}{c_2}\right)^{\frac{1}{2}}, \tag{60}$$

we obtain

$$\lambda_{mn}^{(2)} = c_2(\cot^2\theta_c - \cot^2\theta_{mn}). \tag{61}$$

Table 1 also gives the values of θ_{mn} for the 22 possibly stable steady states. For a given θ_c the steady states with $\theta_{mn} > \theta_c$ are stable. For example, there are 5 stable steady states in the case $\theta_c = 8°$.

Qualitative features of the first two coefficients in the asymptotic series expansion of the bifurcating steady solutions corresponding to the bifurcation points

$$\lambda_{mn} = 26.08 \quad (m = 18, n = 2)$$

and

$$\lambda_{mn} = 12.30 \quad (m = 10, n = 2)$$

are shown in figures 3–6. In order to show the shapes of the interface better, the rigid boundaries are not plotted and in addition both the aspect ratio of the rigid boundaries and the amplitude of the interface are normalized.

We note that the modified second coefficient $2a\eta_{mn}^{(2)}/f_2$ as well as the first coefficient

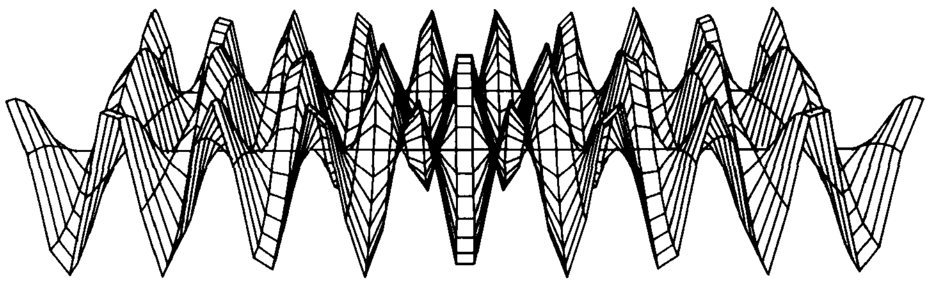

FIGURE 3. Qualitative shape of $\eta^{(1)}_{mn}$ for $\lambda_{mn} = 26.08$ ($m = 18$, $n = 2$).

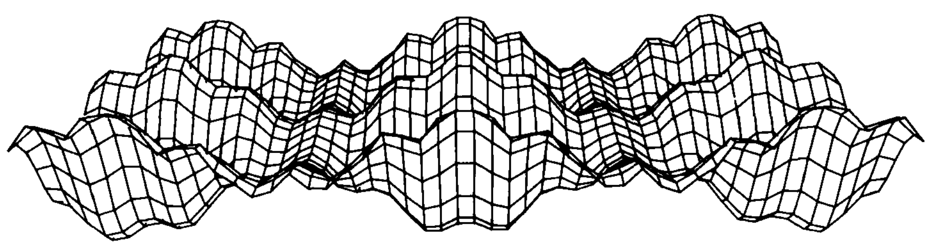

FIGURE 4. Qualitative shape of $2a\eta^{(2)}_{mn}/f_2$ for $\lambda_{mn} = 26.08$ ($m = 18$, $n = 2$).

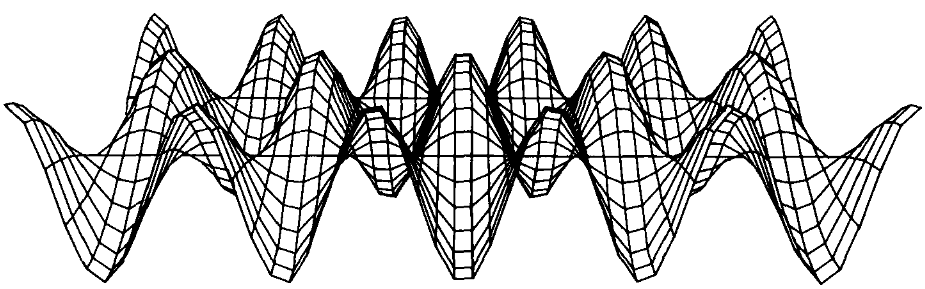

FIGURE 5. Qualitative shape of $\eta^{(1)}_{mn}$ for $\lambda_{mn} = 12.30$ ($m = 10$, $n = 2$).

$\eta^{(1)}_{mn}$ in (45) is independent of the specific form of the interface boundary function f. The extent to which the first-order term in (45) is modified by the second-order term depends not only on the magnitude of the small parameter ϵ but also on the particular interface boundary function f. Figures 7 and 8 show, for $\epsilon = 0.1$ and $\theta_c = 8°$, the shape of the function $\eta^{(1)}_{mn} + \epsilon\eta^{(2)}_{mn}$ corresponding to $\lambda_{mn} = 26.08$, 12.30 respectively. One sees three rows of alternating elevations and depressions parallel to the (x, z)-plane. Lewis's experiment shows that water was penetrated by four air columns at later stages (Lewis 1950, figure 10, profiles 3 and 4). It is rather difficult to imagine what the silhouettes would resemble when the interfaces shown in figures 7 and 8 are projected perpendicularly onto the (x, z)-plane. The calculated shapes do, however, exhibit the interesting features of bubbles-and-spikes configuration.

Before concluding our discussions, it is appropriate to make some remarks about the scope and limitations of the present work. We have set out to see if there exist non-trivial steady states that are more realistic than the initial steady states and to investigate if the presence of rigid boundaries affects the stability analysis. Using a specific class of interface boundary conditions, we have found that non-trivial steady

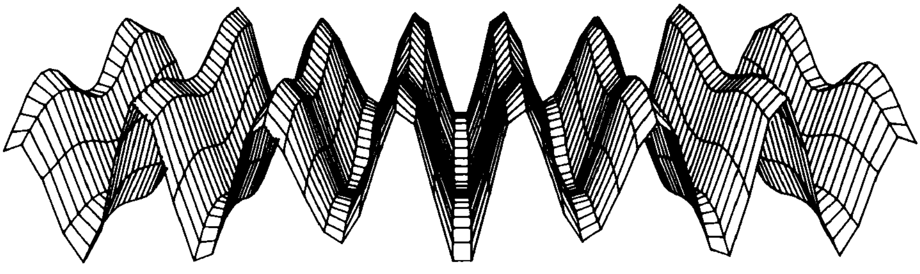

FIGURE 6. Qualitative shape of $2a\eta^{(2)}_{mn}/f_2$ for $\lambda_{mn} = 12.30$ ($m = 10$, $n = 2$).

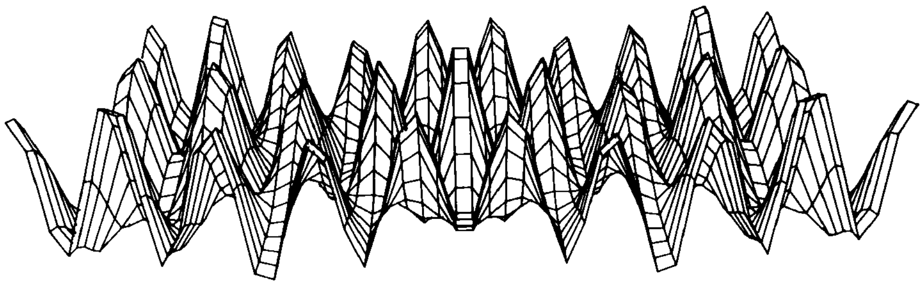

FIGURE 7. Qualitative shape of $\eta^{(1)}_{mn}+\epsilon\eta^{(2)}_{mn}$ for $\lambda_{mn} = 26.08$ ($m = 18$, $n = 2$), $\epsilon = 0.1$ and $\theta_c = 8°$.

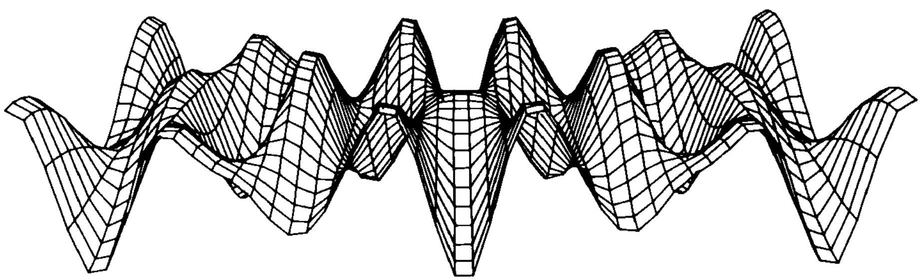

FIGURE 8. Qualitative shape of $\eta^{(1)}_{mn}+\epsilon\eta^{(2)}_{mn}$ for $\lambda_{mn} = 12.30$ ($m = 10$, $n = 2$), $\epsilon = 0.1$ and $\theta_c = 8°$.

states do exist and that some of them can be stable to infinitesimal disturbances. The compromise that allows us to analyse the stability question this way has direct application to problems with more general interface boundary conditions. Moreover, we have considered only one aspect of the rigid-boundary effects, i.e. *to what extent* the presence of rigid boundary affect the stability analysis, and have left out of account other interesting aspects, particularly *in what rigorous manner* can the motion of the fluid in the vicinity of where the interface meets the rigid boundary be modelled.

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Appendix

$$R^{(1)} = 0,$$

$$B_1^{(1)} = B_2^{(1)} = B_3^{(1)} = B_4^{(1)} = 0,$$

$$R^{(2)} = -\lambda_{mn}^{(1)}\,\eta_{mn}^{(1)},$$

$$B_1^{(2)} = -B_2^{(2)} = \frac{f_2}{a}\eta_{mn}^{(1)^2},$$

$$B_3^{(2)} = -B_4^{(2)} = \frac{f_2}{b}\eta_{mn}^{(1)^2},$$

$$R^{(3)} = -a^2b^2(\eta_{mn,y}^{(1)^2}\eta_{mn,xx}^{(1)} - 2\eta_{mn,x}^{(1)}\eta_{mn,y}^{(1)}\eta_{mn,xy}^{(1)} + \eta_{mn,x}^{(1)^2}\eta_{mn,yy}^{(1)})$$
$$-\lambda_{mn}^{(1)}\eta_{mn}^{(2)} - \lambda_{mn}^{(2)}\eta_{mn}^{(1)} - \tfrac{3}{2}\lambda_{mn}\eta_{mn}^{(1)}(a^2\eta_{mn,x}^{(1)^2} + b^2\eta_{mn,y}^{(1)^2}),$$

$$B_1^{(3)} = -B_2^{(3)} = \frac{2f_2}{a}\eta_{mn}^{(1)}\eta_{mn}^{(2)} + \frac{f_3}{a}\eta_{mn}^{(1)^3},$$

$$B_3^{(3)} = -B_4^{(3)} = \frac{2f_2}{b}\eta_{mn}^{(1)}\eta_{mn}^{(2)} + \frac{f_3}{b}\eta_{mn}^{(1)^3}.$$

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